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一类2-指标变延迟微分代数方程BDF方法的收敛性*

刘红良， 肖爱国

(湘潭大学数学与计算科学学院，科学工程计算与数值仿真湖南省重点实验室，湘潭 411105)

摘 要: 延迟微分代数方程经常出现在自动控制、电力和电路分析、多体动力学等许多实际应用问题中．目前对延迟微分代数方程数值分析研究主要集中于线性问题和1-指标问题；对高指标非线性延迟微分代数方程数值分析的研究较困难，国内外仅有少量工作且大多为常延迟．本文将向后微分公式(BDF)应用于求解2-指标非线性变延迟微分代数方程，获得了相应的收敛性结果，并通过数值试验进行了验证．

关键词: 2-指标微分代数方程；向后微分公式；收敛性；变延迟

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1 引言

目前，对延迟微分代数方程的研究思想许多借助了微分代数方程的理论及延迟微分方程的理论，因此常微分方程，微分代数方程及(中立型)延迟微分方程[1,2]等常用的理论及数值方法被许多学者应用于延迟微分代数方程及其数值方法．但是，延迟微分代数方程因有延迟的影响，很多特性是微分代数方程或延迟微分方程所没有的，所以对延迟微分代数方程的研究必须独立进行．Petzold[3]指出“微分代数方程不是常微分方程”，同样，延迟微分代数方程既不是微分代数方程，也不是延迟微分方程．

迄今为止，国内外对延迟微分代数方程的研究集中在理论分析及数值方法的稳定性和误差分析上．在理论分析方面，重点研究了(中立型)延迟(积分)微分代数方程的结构分析及渐近稳定性[4-7]．在数值稳定性方面，重点讨论了Rosenbrock、(多步)Runge-Kutta(RK)方法及线性多步法(含θ-方法、BDF方法)关于线性(中立型)常延迟(积分)微分代数方程的渐近稳定性[4,5,8,9]及θ-方法(含中点法)关于1-指标刚性延迟微分代数方程的稳定性[10,11]．在数值误差分析方面，文献[12-14]分别获得了RK方法、线性多步方法、单支方法及块方法关于1-指标、2-指标常延迟微分代数方程的误差分析结果，文献[15]获得了配置方法关于1-指标、2-指标变延迟半显式非线性微分代数系统的收敛性结果，文献[16,17]获得了单支方法和RK方法关于1-指标变延迟微分代数方程的收敛性结果．

本文应用向后微分公式(BDF)求解一类2-指标非线性变延迟微分代数方程，获得了该方法的收敛性结果，并通过数值试验进行了验证．

收稿日期: 2009-08-05. **作者简介**: 刘红良(1979年11月生)，男，博士. 研究方向：刚性微分方程数值方法.
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2 一类2-指标变延迟微分代数方程的收敛性分析

考虑一类2-指标变延迟微分代数方程(2-DDAEs)

$$\begin{cases} y'(x) = f\big(y(x), y(x-\tau(x)), z(x)\big), & x \in [0,T], \\ 0 = g\big(y(x)\big), & x \in [0,T], \\ z(0) = z_0, \quad y(x) = \varphi(x), & x \in [-\tau, 0], \end{cases} \tag{1}$$

这里延迟函数$\tau(x)$可导，且满足$0 < \tau(x) \le \tau$，$0 < \tau'(x) < 1$，函数$f: R^{n_1} \times R^{n_1} \times R^{n_2} \to R^{n_1}$，$g: R^{n_1} \to R^{n_2}$充分光滑且各阶导数有界，初值函数$\varphi(x)$连续，且$g_y(y) f_z(y, y(x-\tau(x)), z)$可逆．恒设问题(1)有唯一光滑真解$y(x)$，$z(x)$．在下文中符号$\|\cdot\|$表示欧氏范数．

把带p阶插值公式的BDF方法应用于求解问题(1)，可得

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = hf\big(y_{n+k}, y_n^h, z_{n+k}\big), \quad 0 = g(y_{n+k}), \tag{2}$$

这里$x_{n+k} = x_n + kh$，$n \ge 0$，且

$$y_n^h = \begin{cases} \varphi\big(x_{n+k} - \tau(x_{n+k})\big), & x_{n+k} - \tau(x_{n+k}) \le 0, \\ \sum\limits_{j=-u}^{q} Q_j(\delta_n) y_{n+k-m_n+j}, & x_{n+k} - \tau(x_{n+k}) > 0, \end{cases} \tag{3}$$

其中$\tau(x_{n+k}) = (m_n - \delta_n)h$，$u, q, m_n \in Z^+$，$\delta_n \in [0,1)$，$q + u = p$，$q + 1 \le m_n$，$Q_j(\delta_n)$为拉格朗日插值基函数．

考虑(2)的扰动问题

$$\sum_{i=0}^{k} \alpha_i \hat{y}_{n+i} = hf\big(\hat{y}_{n+k}, \hat{y}_n^h, \hat{z}_{n+k}\big) + h\delta, \quad 0 = g(\hat{y}_{n+k}) + \theta. \tag{4}$$

定理1 设y_{n+k}, z_{n+k}由(2)确定，而扰动$\hat{y}_{n+k}, \hat{z}_{n+k}$由(4)确定，起始值满足

$$\begin{aligned} & y_{n+j} - y(x_{n+j}) = O(h), \quad z_{n+j} - z(x_{n+j}) = O(h), \\ & g(y_{n+j}) = O(h^2), \quad j = 0, 1, \cdots, k-1, \quad x_{n+j} = x_n + jh, \end{aligned} \tag{5}$$

及起始值扰动满足

$$\hat{y}_{n+j} - y_{n+j} = O(h^2), \quad \hat{z}_{n+j} - z_{n+j} = O(h), \quad \delta = O(h), \quad \theta = O(h^2), \tag{6}$$

则对$h < h_0$，有

$$\|\hat{y}_{n+k} - y_{n+k}\| \le C\big(\|\hat{Y}_n - Y_n\| + h\|\delta\| + \|\theta\|\big), \tag{7}$$

$$\|\hat{z}_{n+k} - z_{n+k}\| \le \frac{C}{h}\Big(\sum_{j=0}^{k-1} \big\|g_y(\hat{y}_{n+k})(\hat{y}_{n+j} - y_{n+j})\big\| + h\|\hat{Y}_n - Y_n\| + h\|\delta\| + \|\theta\|\Big), \tag{8}$$

这里

$$\hat{Y}_n - Y_n = \big(\hat{y}_{n+k-1}^T - y_{n+k-1}^T, \cdots, \hat{y}_n^T - y_n^T, (\hat{y}_n^h)^T - (y_n^h)^T\big)^T, \tag{9}$$

$$\|\hat{Y}_n - Y_n\| = \max\Big(\max_{0 \le j \le k-1} \big\|\hat{y}_{n+j}^T - y_{n+j}^T\big\|, \big\|(\hat{y}_n^h)^T - (y_n^h)^T\big\|\Big). \tag{10}$$

证明 令

$$\eta=-\sum_{i=0}^{k-1}\frac{\alpha_i}{\alpha_k}y_{n+i},\quad \hat{\eta}=-\sum_{i=0}^{k-1}\frac{\alpha_i}{\alpha_k}\hat{y}_{n+i}. \tag{11}$$

调整 h 和 δ 比例，使得 (2) 和 (4) 分别变为

$$y_{n+k}=\eta+hf(y_{n+k},y_n^h,z_{n+k}),\quad 0=g(y_{n+k}), \tag{12}$$

$$\hat{y}_{n+k}=\hat{\eta}+hf(\hat{y}_{n+k},\hat{y}_n^h,\hat{z}_{n+k})+h\delta,\quad 0=g(\hat{y}_{n+k})+\theta. \tag{13}$$

(12) 中第二式可改写为

$$0=g(y_{n+k})-g(\eta)+g(\eta)=\int_0^1 g_y\big(\eta+\tau(y_{n+k}-\eta)\big)\mathrm{d}\tau[y_{n+k}-\eta]+g(\eta), \tag{14}$$

把 (12) 中第一式代入 (14) 可得

$$\int_0^1 g_y\big(\eta+\tau(y_{n+k}-\eta)\big)\mathrm{d}\tau f(y_{n+k},y_n^h,z_{n+k})+\frac{1}{h}g(\eta)=0, \tag{15}$$

同理，(13) 中第二式改写为

$$\int_0^1 g_y\big(\hat{\eta}+\tau(\hat{y}_{n+k}-\hat{\eta})\big)\mathrm{d}\tau\left[f(\hat{y}_{n+k},\hat{y}_n^h,\hat{z}_{n+k})+\delta\right]+\frac{1}{h}g(\hat{\eta})+\frac{1}{h}\theta=0. \tag{16}$$

利用函数 f,g 的光滑性及 g_yf_z 的可逆性，由 (15)-(16) 可得

$$\begin{aligned}\|\hat{z}_{n+k}-z_{n+k}\| \le\ & C_1\Big(\|\hat{y}_{n+k}-y_{n+k}\|+\|\hat{\eta}-\eta\|+\|\hat{y}_n^h-y_n^h\|\\ &+\|\delta\|+\frac{1}{h}\|\theta\|+\frac{1}{h}\|g(\hat{\eta})-g(\eta)\|\Big),\end{aligned} \tag{17}$$

由 (12) 中第一式减去 (13) 中第一式可得

$$\begin{aligned}\|\hat{y}_{n+k}-y_{n+k}\| &\le \|\hat{\eta}-\eta\|+h\big\|f(\hat{y}_{n+k},\hat{y}_n^h,\hat{z}_{n+k})-f(y_{n+k},y_n^h,z_{n+k})\big\|\\ &\le \|\hat{\eta}-\eta\|+hL\big(\|\hat{y}_{n+k}-y_{n+k}\|+\|\hat{y}_n^h-y_n^h\|+\|\hat{z}_{n+k}-z_{n+k}\|\big),\end{aligned} \tag{18}$$

这里 L 为右端函数 f 的经典 Lipschitz 常数，把 (17) 代入 (18)，化简得

$$\|\hat{y}_{n+k}-y_{n+k}\|\le C_2\big(\|\hat{\eta}-\eta\|+\|\hat{y}_n^h-y_n^h\|+h\|\delta\|+\|\theta\|\big),\quad h\le\frac{1}{L+LC_1}. \tag{19}$$

把 (19) 代入 (17)，化简得

$$\begin{aligned}\|\hat{z}_{n+k}-z_{n+k}\| \le\ & C_3\Big(\|\hat{\eta}-\eta\|+\|\hat{y}_n^h-y_n^h\|+\|\delta\|\\ &+\frac{1}{h}\|\theta\|+\frac{1}{h}\|g_y(\hat{\eta})(\hat{\eta}-\eta)\|\Big),\quad h\le\frac{1}{L+LC_1},\end{aligned} \tag{20}$$

又

$$\|\hat{\eta}-\eta\|=\Big\|\sum_{j=0}^{k-1}\frac{\alpha_j}{\alpha_k}(\hat{y}_{n+j}-y_{n+j})\Big\|\le C_4\|\hat{Y}_n-Y_n\|, \tag{21}$$

$$\begin{aligned}\big\|g_y(\hat{\eta})(\hat{\eta}-\eta)\big\| &\le \big\|g_y(\hat{y}_k)(\hat{\eta}-\eta)\big\|+\|\hat{\eta}-\eta\|O(h)\\ &\le C_5\Big(\sum_{j=0}^{k-1}\big\|g_y(\hat{y}_k)(\hat{y}_{n+j}-y_{n+j})\big\|+\|\hat{Y}_n-Y_n\|O(h)\Big).\end{aligned} \tag{22}$$

把 (21), (22) 代入 (19), (20)，经化简得结论 (7), (8).

推论1 设2-DDAEs (1) 满足 $g_y f_z$ 可逆，且带 p 阶插值公式 (3) 的BDF方法 (2) 是 p 阶相容的，则它的局部截断误差满足

$$y_k - y(x_k) = O(h^{p+1}), \quad z_k - z(x_k) = O(h^p). \tag{23}$$

证明 令定理1中 $n=0$, $\hat{y}_j = y(x_j)$, $\hat{z}_j = z(x_j)$, $j = 0,1,\cdots,k$，这些值满足定理1的条件，且 $\delta = O(h^p)$, $\theta = 0$，由插值公式 (3) 知 $\|\hat{y}_0^h - y_0^h\| = O(h^{p+1})$，应用定理1即得结论.

定理2 设2-DDAEs (1) 满足 $g_y f_z$ 可逆，$k \le 6$，方法 (2) 起始值满足

$$y_j - y(x_j) = O(h^{p+1}), \quad j = 0,1,\cdots,k-1, \tag{24}$$

则带 p 阶插值公式 (3) 的 k 步BDF方法 (2) 是 $p=k$ 阶收敛的，即

$$y_n - y(x_n) = O(h^p), \quad z_n - z(x_n) = O(h^p), \quad x_n = nh, \quad n \ge k. \tag{25}$$

证明 首先研究 y 分量的局部误差积累. 数值解序列记为 $\{y_n^0, z_n^0\}$，考虑多步解序列 $\{y_n^l, z_n^l\}$, $l = 1,2,\cdots$，起始值为 $y_j^l = y(x_j)$, $z_j^l = z(x_j)$, $j = l-1,\cdots,l+k-2$. 首要任务是估计 $y_n^l - y_n^{l+1}$，为了方便起见，我们去掉上标，两个相邻多步解分别记为 $\{\hat{y}_n, \hat{z}_n\}$ 和 $\{\tilde{y}_n, \tilde{z}_n\}$. 为了能利用定理1的结论，固定3个充分大的常数 $\hat{C}_0$, $\hat{C}_1$, $\hat{C}_2$ (文后注释说明了这3个常数的合理性)，使

$$\begin{aligned}&\left\|\hat{y}_{n+j} - y(x_{n+j})\right\| \le \hat{C}_0 h, \quad \left\|\hat{z}_{n+j} - z(x_{n+j})\right\| \le \hat{C}_1 h,\\ &\left\|\hat{y}_{n+j} - \tilde{y}_{n+j}\right\| \le \hat{C}_2 h^2, \quad j = 0,1,\cdots,k-1.\end{aligned} \tag{26}$$

引入记号 $\Delta z_{n+k} = \tilde{z}_{n+k} - \hat{z}_{n+k}$, $\Delta y_{n+j} = \tilde{y}_{n+j} - \hat{y}_{n+j}$, $j = 0,1,\cdots,k$, $\Delta y_n^h = \tilde{y}_n^h - \hat{y}_n^h$, $\Delta Y_n = (\Delta y_{n+k-1}^T, \cdots, \Delta y_n^T, (\Delta y_n^h)^T)^T$，在定理1中取 $\delta = 0$, $\theta = 0$，应用定理1可得

$$\|\Delta y_{n+k}\| \le C\big(\|\Delta Y_n\|\big), \quad \|\Delta z_{n+k}\| \le \frac{C}{h}\Big(\sum_{j=0}^{k-1}\left\|g_y(\hat{y}_{n+k})\Delta y_{n+j}\right\| + h\|\Delta Y_n\|\Big). \tag{27}$$

只要 h 充分小时，常数 C 不依赖于 $\hat{C}_0$, $\hat{C}_1$, $\hat{C}_2$ 的选取. 由插值公式 (3) 及假设 (26)，有

$$\|\Delta y_{n+k}\| = O(h^2), \quad \|\Delta z_{n+k}\| = O(h). \tag{28}$$

线性化多步公式得

$$\begin{aligned}\sum_{i=0}^{k}\alpha_i \Delta y_{n+i} &= h\big[f(\tilde{y}_{n+k}, \tilde{y}_n^h, \tilde{z}_{n+k}) - f(\hat{y}_{n+k}, \hat{y}_n^h, \hat{z}_{n+k})\big]\\ &= h f_z(\hat{y}_{n+j}, \hat{y}_n^h, \hat{z}_{n+j})\Delta z_{n+j} + O\big(h\|\Delta Y_n\|\big),\end{aligned} \tag{29}$$

$$0 = g_y(\hat{y}_{n+k})\Delta y_{n+k} + O\big(h\|\Delta Y_n\|\big). \tag{30}$$

记 $Q_{n+j} = (f_z(g_y f_z)^{-1} g_y)(\hat{y}_{n+j}, \hat{y}_n^h, \hat{z}_{n+j})$, $P_{n+j} = I - Q_{n+j}$, $j = 0,1,\cdots,k$，易证

$$Q_{n+j}^2 = Q_{n+j}, \quad P_{n+j}^2 = P_{n+j}, \quad Q_{n+j}P_{n+j} = P_{n+j}Q_{n+j} = 0, \quad Q_{n+j+1} = Q_{n+j} + O(h), \tag{31}$$

用 P_{n+k} 左乘 (29)，$f_z(g_y f_z)^{-1}$ 左乘 (30)，可得

$$\sum_{i=0}^{k} \alpha_i P_{n+i} \Delta y_{n+i} = O\big(h\|\Delta Y_n\|\big), \quad Q_{n+k}\Delta y_{n+k} = O\big(h\|\Delta Y_n\|\big). \tag{32}$$

由插值公式 (3)，延迟函数 $0 < \tau'(x) < 1$ 可导，假设 (26) 及 (27)，有

$$\frac{1}{2}\Delta y_{n+1}^h = \frac{1}{2}\Delta y_n^h + O\big(h\|\Delta Y_n\|\big), \tag{33}$$

记

$$U_n = \Big(\big(P_{n+k-1}\Delta y_{n+k-1}\big)^T, \cdots, \big(P_n \Delta y_n\big)^T, \frac{1}{2}\big(\Delta y_n^h\big)^T\Big)^T,$$

$$V_n = \Big(\big(Q_{n+k-1}\Delta y_{n+k-1}\big)^T, \cdots, \big(Q_n \Delta y_n\big)^T, \frac{1}{2}\big(\Delta y_n^h\big)^T\Big)^T,$$

有 $\Delta Y_n = U_n + V_n$，(32) 式可分别化为

$$U_{n+1} = (A \otimes I)U_n + O\big(h\|U_n\| + h\|V_n\|\big), \tag{34}$$

$$V_{n+1} = (N \otimes I)V_n + O\big(h\|U_n\| + h\|V_n\|\big), \tag{35}$$

这里

$$A = \begin{pmatrix} -\alpha'_{k-1} & \cdots & -\alpha'_1 & -\alpha'_0 & 0 \\ 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 \\ 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix},$$

其中 $\alpha'_j = \alpha_j/\alpha_k$. 由于 $\rho(\xi) = \sum\limits_{j=0}^{k} \alpha_k \xi^k$ 满足根条件，类似于文献 [18]，可选取一范数 $\|\cdot\|$，使得 $\|A \otimes I\| \le 1$，及另一可能不同的范数 $\|\cdot\|$，使得 $\|N \otimes I\| \le 1$，因此有

$$\begin{pmatrix} \|U_{n+1}\| \\ \|V_{n+1}\| \end{pmatrix} \le \begin{pmatrix} 1+O(h) & O(h) \\ O(h) & 1+O(h) \end{pmatrix} \begin{pmatrix} \|U_n\| \\ \|V_n\| \end{pmatrix}, \tag{36}$$

化 (36) 中系数矩阵成对角阵，易得

$$\begin{pmatrix} \|U_n\| \\ \|V_n\| \end{pmatrix} \le T^{-1} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} T \begin{pmatrix} \|U_0\| \\ \|V_0\| \end{pmatrix}, \tag{37}$$

这里 $\lambda_1 = 1 + O(h)$, $\lambda_2 = 1 + O(h)$，变换矩阵 T 由相应特征向量构成，满足

$$T = \begin{pmatrix} 1 & O(h) \\ O(h) & 1 \end{pmatrix},$$

化简得

$$\|U_n\| \le \check{C}\big(\|U_0\| + h\|V_0\|\big), \tag{38}$$

$$\|V_n\| \le \check{C}\big(h\|U_0\| + \|V_0\|\big), \tag{39}$$

$$\|Y_n\| \le C_6\big(\|U_n\| + \|V_n\|\big) \le C_7\big(\|U_0\| + \|V_0\|\big). \tag{40}$$

向量 U_0, V_0 由 y 变量的起始值的误差及局部截断误差组成，满足

$$\|U_0\| \le H_0 h^{p+1}, \quad \|V_0\| \le H_1 h^{p+1}, \quad \|\Delta y_0\| \le H_2 h^{p+1}, \tag{41}$$

由 (23), (24), (38)-(40) 及 (41)，可知

$$\|\Delta y_n\| \le C_8 h^{p+1}, \quad \big\|g_y(\hat{y}_{n+k})\Delta y_{n+j}\big\| \le C_9 h^{p+1}, \tag{42}$$

$$\|y_n - y(x_n)\| \le \sum_{l=0}^{n-k+1} \|y_n^l - y_n^{l+1}\| \le C_{10} h^p. \tag{43}$$

由于 z_n 仅依赖于 $y_{n-k}^h, y_{n-k}, \cdots, y_{n-1}$，在定理 1 中取 $\hat{y}_i = y(x_i)$, $\hat{z}_i = z(x_i)$, $\delta = O(h^p)$, $\theta = 0$，应用定理 1 的结论可得

$$\|z_n - z(x_n)\| \le \frac{C}{h}\sum_{j=0}^{k} \big\|g_y(y(x_n))(y_{n-j} - y(x_{n-j}))\big\| + O(h^p). \tag{44}$$

利用 (42) 和 (43)，可得

$$\begin{aligned}\big\|g_y(y(x_n))(y_{n-j} - y(x_{n-j}))\big\| &= \sum_{l=0}^{n-k+1} \big\|g_y(y_n^l + O(h^p))(y_{n-j}^l - y_{n-j}^{l+1})\big\| \\ &\le \sum_{l=0}^{n-k+1} \Big(\big\|g_y(y_n^l)(y_{n-j}^l - y_{n-j}^{l+1})\big\| + O(h^{2p+1})\Big) = O(h^{p+1}),\end{aligned}$$

因此

$$\big\|z_n - z(x_n)\big\| \le C_{11} h^p. \tag{45}$$

注 1 一般来说，常数 C_{10}, C_{11} 依赖于假设 (26) 中的 $\hat{C}_0, \hat{C}_1$，我们严格限制步长 h，使得 $C_{10}h^{p-1} \le \hat{C}_0$, $C_{11}h^{p-1} \le \hat{C}_1$，这样，数值解就不会与假设 (26) 相冲突.

3 数值试验

考虑如下 2-指标变延迟微分代数初值问题

$$\begin{cases} y_1'(x) = -2\sqrt{y_1 z} + y_2 \frac{y_1(\frac{x}{2})}{y_2(\frac{x}{2})}, & 0 \le x \le 2, \\ y_2'(x) = -\frac{z^2}{y_2} - y_1 y_2(\frac{x}{2}) - \sqrt{z^3}, & 0 \le x \le 2, \\ 0 = y_1 y_2 + y_2^2, & 0 \le x \le 2, \\ y_1(0) = 1, \quad y_2(0) = -1, \quad z(0) = 1, \end{cases} \tag{46}$$

问题(46)有唯一真解$y_1(x)=\mathrm{e}^{-x}$，$y_2(x)=-\mathrm{e}^{-x}$，$z(x)=\mathrm{e}^{-x}$．取步长h，y_1，y_2，z在$x=2$处的整体误差分别记为$yerr1(h)$，$yerr2(h)$，$zerr(h)$．通过计算可以得到相应的误差阶，分别记为$py1(h)$，$py2(h)$，$pz(h)$，其中

$$py1(h)=\ln\frac{yerr1(h)}{yerr1(0.5h)}\Big/\ln 2,\quad py2(h)=\ln\frac{yerr2(h)}{yerr1(0.5h)}\Big/\ln 2,\quad pz(h)=\ln\frac{zerr(h)}{zerr(0.5h)}\Big/\ln 2.$$

分别用带线性插值公式的二步二阶BDF方法(BDF2)和带二次插值公式的三步三阶BDF方法(BDF3)求解初值问题(46)．数值结果如表1和表2所示，验证了理论结果的正确性．

表1: BDF2的数值结果

h	$yerr1$	$yerr2$	$zerr$	$py1$	$py2$	pz	CPU
0.1	$0.3590E-2$	$0.4899E-2$	$0.7199E-2$	1.9974	2.0283	2.0865	0.1590
0.05	$0.8991E-3$	$0.1201E-2$	$0.1695E-2$	2.0031	1.9750	1.9924	0.3167
0.025	$0.2243E-3$	$0.3055E-3$	$0.4260E-3$	2.0019	1.9976	2.0122	0.6085
0.0125	$0.0560E-3$	$0.0765E-3$	$0.1056E-3$				1.1973

表2: BDF3的数值结果

h	$yerr1$	$yerr2$	$zerr$	$py1$	$py2$	pz	CPU
0.1	$0.5100E-3$	$0.7001E-3$	$0.1100E-2$	2.9521	2.9436	3.0179	0.2179
0.05	$0.6590E-4$	$0.9100E-4$	$0.1358E-3$	2.9684	3.0019	3.0541	0.4064
0.025	$0.8420E-5$	$0.1136E-4$	$0.1635E-4$	3.0076	2.9501	3.0589	0.7932
0.0125	$0.1047E-5$	$0.1470E-5$	$0.1962E-5$				1.4571

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Convergence of Backward Differentiation Formulas for Index-2 Differential-algebraic Equations with Variable Delay

LIU Hong-liang, XIAO Ai-guo

(School of Mathematics and Computational Science, Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Xiangtan 411105)

Abstract: Delay differential-algebraic equations often arise in automatic control, power and circuit analysis, multi-body dynamics, etc. The current researches on numerical analysis for delay-differential-algebraic equations are mainly focused on linear problems and 1-index problems. It is difficult to do numerical analysis for high-index nonlinear delay-differential-algebraic equations, and there are only a few results for this kind of problems, furthermore, most of them are about constant-delay problems. The backward differentiation formulas are applied in this paper to index-2 nonlinear differential-algebraic equations with variable delay. The corresponding convergence results are obtained and confirmed by some numerical examples.

Keywords: index-2 differential-algebraic equations; backward differentiation formulas; convergence; variable delay

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